0-CYCLES AND SHEAVES ON ABELIAN SURFACES

GIOVANNI MONGARDI, GIANLUCA PACIENZA, AND LAURA PERTUSI

Abstract. We introduce a filtration on the Chow ring of an abelian surface A, inspired by O'Grady's filtration on K3 surfaces. We give a geometric description of the filtration, and we prove that it is deeply linked with the rational orbit of points in the generalized Kummer variety of A. We propose a conjecture on the second Chern class of sheaves on this abelian surface, and we provide some evidence for the conjecture and prove some of its applications.

1. Introduction

On a projective K3 surface S, Beauville and Voisin [BV04] proved that there is a canonical zero cycle o_S, given by the class of any point on any rational curve on S, such that for every divisors D, D' one has $D \cdot D' \in \mathbb{Z} o_S$. One can define a filtration in the Chow ring of zero cycles of S, where $S_g(S)$ is given by cycles equivalent to $x_1 + \dots x_g + k o_S$, with $k \in \mathbb{Z}$. O'Grady [O'13] proved that $S_g(S)$ consists of cycles supported on genus g curves on S. Building upon O'Grady's work and generalizing a result by Huybrechts [Huy10], Voisin [Voi15] proved that the second Chern class of simple vector bundles falls into $S_d(S)$, where d is half the dimension of the moduli space containing the vector bundle. By considering stable objects in S, Shen Yin and Zhao [SYZ20] greatly generalized Voisin's result and proved that the second Chern class of a stable object lies in the prescribed piece of the above filtration.

The goal of our work is to introduce a filtration on the Chow group $\mathrm{CH}_0(A)$ of 0-cycles on an abelian surface A. To avoid ambiguities we denote by $a +_A b$ the sum, with respect to the group structure, of two points $a, b \in A$ on an abelian surface A and by $a + b$ their sum in the Chow group $\mathrm{CH}_0(A)$. Inspired by the above works on projective $K3$ surfaces, for any positive integer d we define the following subset of $\mathrm{CH}_0(A)$ which will turn out to define an increasing filtration which is stable under multiplication by $\mathbb{Z}$ (cf. Lemma 3.3 and Corollary 3.5):

$$S_d(A) := \{z \in \mathrm{CH}_0(A) : z + \iota(z) \equiv z' + 2m o_A \text{ with } m \in \mathbb{Z} \text{ and } z' \text{ effective of degree } 2d\},$$

where $o_A \in A$ is the origin, ι the (-1)-involution and $\equiv$ the rational equivalence relation. For any integer $k \geq d$ we define inside $S_d(A)$ the following

$$S_d^k(A) := \{z \in \mathrm{CH}_0(A) : z + \iota(z) \equiv z' + 2(k-d) o_A \text{ with } m \in \mathbb{Z} \text{ and } z' \text{ effective of degree } 2d\}.$$

The filtration above can again be interpreted as cycles supported on curves (cf. Lemma 3.4).

Let $z \in A^{(d)}$. We denote by O_z^ι the set of all points $w \in A^{(d)}$ such that

$$z + \iota(z) \equiv w + \iota(w) \in \mathrm{CH}_0(A)$$

and by O_z the set of all points $w \in A^{(d)}$ such that $z \equiv w \in \mathrm{CH}_0(A)$.

Theorem 1.1. *Let A be an abelian surface and $k > d \geq 0$ two integers. Then*

$$S_d^k(A) = \{z \in \mathrm{CH}_0(A) : O_z^\iota \neq \emptyset \text{ and } \dim O_z^\iota \geq (k-d)\}.$$

We then make the following conjecture:

Conjecture 1.2. *Let A be an abelian surface and F a simple vector bundle on A with Mukai vector $v := v(F)$. Suppose that $c_1(F)$ is symmetric. Then $c_2(F) \in S_{d(v)-1}(A)$.*

2020 *Mathematics Subject Classification.* 14J42, 14K12 14C25, 14F06.

Recall that given a Mukai vector $v = (r, l, s)$, we set $d(v) := \frac{l^2}{2} - rs + 1$. This number coincides with $\frac{1}{2}\dim(\mathrm{Ext}^1(F, F))$ for any stable sheaf F with Mukai vector v, that is half the dimension of the moduli space $M_H(v)$ (for any v generic polarization H).

We provide partial evidence for this conjecture in Theorem 5.2. In Section 6, we prove that the above conjecture implies the same for all symmetric stable torsion free sheaves (cf. Theorem 6.1).

An important consequence of the above conjecture concerns the existence of the coisotropic subvarieties as predicted by Voisin (cf. [Voi16]). More precisely, denote by $K_H(v)$ the fiber of the Albanese morphism of the moduli space $M_H(v)$ of semistable sheaves with respect to a v-generic polarization H on A and Mukai vector v.

Theorem 1.3. *Let A be an abelian surface and v a primitive positive Mukai vector with $v^2 \geq 6$. Let H be a v-generic polarization on A. Assume that Conjecture 1.2 holds. Then for any $0 < i \leq d(v) - 1$ there exists an algebraically coisotropic subvariety $Z \dashrightarrow B$ of codimension $i - 1$ in $K_H(v)$ with constant cycle fibers.*

While this paper was being prepared, Chen, Li and Zhang posted the very interesting work [CLZ26] on the arXiv. Among other things they introduce a similar framework to study zero cycles on generalized Kummer type manifolds. We compare [CLZ26, Conjecture 7.2] with Conjecture 1.2 at the end of Section 7.

Acknowledgements. We would like to thank Kieran O'Grady and Xiaolei Zhao for many useful comments on a preliminary version of this work and Zaiyuan Chen, Zhiyuan Li, and Ruxuan Zhang for kind email exchange.

G.P. is partially supported by ANR project No. ANR-23-CE40-0026 (POK0). L.P. is supported by the FIS 3 Starting Grant Stability Conditions on Noncommutative Varieties with Applications in Algebraic Geometry (FIS-2024-04715), and is a member of the Indam group GNSAGA.

2. Symmetric curves and cycles on abelian surfaces

In this section we prove that there are curves in any abelian surface which are symmetric and cover a curve of given genus g on the quotient K3 surface S. In the following, a ι-invariant curve/cycle/line bundle will be called symmetric. We denote with $\mathrm{Pic}^\iota(A)$ the group of symmetric line bundles. Notice that, by [BL13, Lemma 4.6.2], for any $l \in \mathrm{NS}(A)$ there exists a symmetric line bundle L such that $c_1^{top}(L) = l$.

Proposition 2.1. *Let A be an abelian surface and let S be the minimal resolution of the quotient A/ι. Then, for any given genus g, there exists an integral curve C of geometric genus g inside S whose double cover $\widetilde{C} \subset A$ is a symmetric curve.*

Proof. It suffices to prove the existence of the curve C on S, as the double cover will naturally be symmetric. In this setting, our claim is precisely [CGL22, Theorem A]. □

Remark 2.2. An interesting case is when $g = 0$. We will call symmetric hyperelliptic all curves on A covering an integral rational curve on S. The name is motivated by the fact that the curve is ι-invariant and the action of ι on it coincides with a hyperelliptic involution. In particular, by the above proposition there exist ample and irreducible symmetric hyperelliptic curves. Moreover, as [CGL22, Theorem A] gives a Zariski dense set of rational curves in S, it proves also the Zariski density of symmetric hyperelliptic curves on A. Furthermore, notice that for any point p on a symmetric hyperelliptic curve we have $p + \iota(p) \equiv 2o_A$: indeed by definition the class of any element of the g^1_2 given by the hyperelliptic involution ι is equivalent to the pullback of the class of the point o_S, which is equal to $2o_A$.

The following proposition is contained in [Bea86], we provide here a different proof relying on the previous result:

Proposition 2.3. *The image of the intersection map restricted to symmetric classes*

$$\mathrm{Pic}(A)^{\iota} \otimes Pic(A)^{\iota} \longrightarrow \mathrm{CH}_0(A),\ L \otimes M \mapsto L \cdot M.$$

has image in $2\mathbb{Z}o_A$.

Proof. As $\mathrm{Pic}(A)^{\iota}$ is generated by ample symmetric classes, it is sufficient to check that

$$H \cdot H' \in \mathbb{Z}o_A \tag{2.1}$$

for H, H' ample symmetric line bundles. By Proposition 2.1, there exist $C \in |H|$, $C' \in |H'|$ symmetric hyperelliptic curves. In particular for any $p \in C \cap C'$ we have $\iota(p) \in C \cap C'$. Hence

$$C \cap C' = \sum_{i=1}^{m} p_i + \sum_{i=1}^{m} \iota(p_i)$$

from which we deduce

$$H \cdot H' \equiv C \cdot C' \equiv 2m \cdot o_A \tag{2.2}$$

by Remark 2.2. □

We now prove a result which will be useful further on. The analog for general $K3$ surfaces was proved in [Mac04, Theorem 1.2] and its statement and proof inspired our result.

Theorem 2.4. *Let A be an abelian surface and $x \in A$ a general point. Then*

$$\{y \in A : y + \iota(y) \equiv x + \iota(x)\}$$

is dense for the euclidean topology in A.

Proof. We will be using [Mac04, Theorem 1.2 and its proof]: the key point in Maclean's proof is the existence of two distinct 1-parameter families of irreducible elliptic curves on a general $K3$ surface. We use [CGL22, Theorem A], which gives us the existence of (infinitely many) integral elliptic curves on S, which move in a one dimensional family by [CGL22, Proposition 2.9]. Hence Maclean's result [Mac04, Theorem 1.2] holds on S, so that for any $p \in S$ general the locus $\{q \in S : q \equiv p\}$ is dense for the euclidean topology on S. It suffices now to take any preimage $y \in A$ of p and by pullback we have

$$\{\iota^*(p) \equiv \iota^*(q)\} = \{y \in A : y + \iota(y) \equiv x + \iota(x)\}$$

is dense for the euclidean topology on A.

□

Remark 2.5. We remark that, using [CGL22, Theorem A and Proposition 2.9] one can actually prove [Mac04, Theorem 1.2] for any complex K3 surface S, so that for general $x \in S$ the locus

$$\{y \in S \text{ such that } y \equiv x\}$$

is dense in the Euclidean topology.

3. The filtration on CH_0

Following [O'13] and [Voi15], we define the following filtration on the Chow ring of an abelian surface:

Definition 3.1. Let A be an abelian surface and $d \geq 0$ an integer. We set

$$S_d(A) := \{z \in \mathrm{CH}_0(A) : z + \iota(z) \equiv y + 2m \cdot o_A,\ m \in \mathbb{Z},\ y \text{ effective 0-cycle of degree } 2d\}.$$

For any $k \geq d$ we will consider $S_d^k(A) \subset S_d(A)$ defined as follows

$$S_d^k(A) := \{z \in \mathrm{CH}_0(A) : z + \iota(z) \equiv y + 2(k-d) \cdot o_A,\quad y \text{ effective 0-cycle of degree } 2d\}.$$

Notice that, by construction, the cycle y in the above definition is symmetric.

Remark 3.2. By definition we have that if $z \in S_d(A)$ and $z' \in S_{d'}(A)$, then $z + z' \in S_{d+d'}(A)$.

Lemma 3.3. *$S_d(A)$ defines an increasing filtration $S_\bullet(A)$ on $\mathrm{CH}_0(A)$.*

Proof. Consider $z \in \mathrm{CH}_0(A)$ such that $z + \iota(z) \equiv y + 2m \cdot o_A$, $m \in \mathbb{Z}$, y effective 0-cycle of degree $2d$. Let $p = p_{d+1}$ be any point on a symmetric hyperelliptic curve on A, which exists by Proposition 2.1. Then $p + \iota(p) \equiv 2o_A$. Hence

$$z + \iota(z) \equiv \big(y + p + \iota(p)\big) + 2(m-1) \cdot o_A$$

which shows the inclusion $S_d(A) \subset S_{d+1}(A)$, $\forall d \geq 0$. □

We give a characterization of the filtration $S_\bullet(A)$ in terms of cycles supported on symmetric curves covering a curve of given genus.

Lemma 3.4. *For any $g \geq 0$ we have*

$$S_g(A) = \bigcup \{f_* \mathrm{CH}_0(\widetilde{C})\},$$

where $f : \widetilde{C} \longrightarrow A$ is a non-constant map and $\widetilde{C}$ is a smooth curve such that $f(\widetilde{C})$ is symmetric and the induced involution ι on $\widetilde{C}$ is such that $C = \widetilde{C}/\iota$ has genus g.

Proof. We prove the inclusion $\supset$. Let $\widetilde{C}$ be a smooth curve and $f : \widetilde{C} \longrightarrow A$ a non-constant map such that $f(\widetilde{C})$ is symmetric. By Proposition 2.1, such a curve exists and the quotient curve $C := \widetilde{C}/\iota$ of genus g has a non constant map to the K3 surface $S = \widetilde{A}/\iota$. Let $\pi : \widetilde{C} \longrightarrow C$ be the quotient map. Let $z \in f_* \mathrm{CH}_0(\widetilde{C})$. If $g = 0$ by definition $z + \iota(z) = 2\deg(z) o_A$, as the hyperelliptic involution coincides with ι. We may then assume $g \geq 1$. Let D be an ample symmetric hyperelliptic curve as provided by Remark 2.2. By the invariance of $\widetilde{C}$ and D, for any $q \in \widetilde{C} \cap D$ we have $\iota(q) \in \widetilde{C} \cap D$, hence $q + \iota(q) \equiv 2o_A$. Consider the following 0-cycle of degree g:

$$\xi := z - (\deg(z) - g)q$$

and let $\eta \in \mathrm{CH}_0(C)$ be a cycle such that $\xi + \iota(\xi) = \pi^*(\eta)$. Notice that η still has degree g. By Riemann-Roch on C we have

$$h^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(\xi + \iota(\xi))) \geq h^0(C, \mathcal{O}_C(\eta)) = \deg(\eta) + 1 - g = 1 > 0. \tag{3.1}$$

So, up to linear equivalence, we can suppose η is effective and so is $\xi + \iota(\xi)$. Hence $z + \iota(z)$ can be written as

$$z + \iota(z) \equiv \big(\xi + \iota(\xi)\big) + (\deg(z) - g)q + (\deg(z) - g)\iota(q) \equiv \big(\xi + \iota(\xi)\big) + 2(\deg(z) - g)o_A$$

with $\xi + \iota(\xi)$ a degree $2g$ cycle which is effective by (3.1).

Let us now prove the other inclusion: let $z \in S_g(A)$ and write

$$z + \iota(z) = p_1 + \ldots p_g + \iota(p_1) + \cdots + \iota(p_g) + 2m \cdot o_A.$$

Let q_i be the projection of p_i to the K3 surface S[1]. By a parameter count, there exists a curve C of genus at most g passing through the points $q_1, \ldots q_g$, and let $f : \widetilde{C} \longrightarrow \iota^{-1}(C)$ be a resolution of singularities. By construction $f(\widetilde{C})$ contains the points $p_1, \ldots p_g$ and $\iota(p_1), \ldots, \iota(p_g)$, so that $z \in f_* CH_0(\widetilde{C})$. □

Corollary 3.5. *For all $d \geq 0$ we have $\mathbb{Z} \cdot S_d(A) \subset S_d(A)$.*

Proof. This follows from the elementary observation that $\mathbb{Z} \cdot \mathrm{CH}_0(C) \subset \mathrm{CH}_0(C)$ for any curve C and from the previous Lemma 3.4. □

[1]Notice that such a projection is not well defined if one of the points is of two torsion. However, in the crepant resolution $S \longrightarrow A/\iota$ all points over the image of a two torsion point are rationally equivalent, hence we can pick any of them.

4. A characterization of the filtration via rational orbits

Let $z \in A^{(d)}$. We denote by O^{ι}_z the set of all points $w \in A^{(d)}$ such that $z + \iota(z) \equiv w + \iota(w) \in \mathrm{CH}_0(A)$. We start with the following.

Lemma 4.1. *Let A be an abelian surface and $k > d \geq 0$ two integers. Then*

$$S^k_d(A) \subset \{z \in \mathrm{CH}_0(A) : O^{\iota}_z \neq \emptyset \text{ and } \dim O^{\iota}_z \geq (k-d)\}.$$

Proof. Let $C \subset A$ be a symmetric hyperelliptic curve. For any $p \in C$, then $p + \iota(p) \equiv 2o_A$, and therefore for any $z \in C^{(k-d)}$, the cycle $z + \iota(z)$ is contained in the orbit of $2(k-d)o_A$. As a consequence, any 0-cycle $z \in \mathrm{CH}_0(A)$ such that $z + \iota(z) \equiv y + 2(k-d)o_A$ with y effective 0-cycle of degree k verifies $O^{\iota}_z \neq \emptyset$ and $\dim O^{\iota}_z \geq (k-d)$. □

We also record the two following results.

Lemma 4.2. *Let A be an abelian surface and $C \subset A$ a symmetric curve such that $\forall p, q \in C$ verify*

$$p + \iota(p) \equiv q + \iota(q) \tag{4.1}$$

in A. Then $\forall p \in C$ we have $p + \iota(p) \equiv 2o_A$.

Proof. Take an ample symmetric hyperelliptic curve $D \subset A$ and consider $C \cap D$. For any $p \in C \cap D$ we have $\iota(p) \in C \cap D$, hence $p + \iota(p) \equiv 2o_A$ by Remark 2.2, and the conclusion follows from (4.1). □

Lemma 4.3. *Let A be an abelian surface. The union of curves C satisfying the property stated in Lemma 4.2 is Zariski dense in A.*

Proof. By [Voi15, Lemma 2.3] applied to the Kummer $K3$ surface S associated to A, the union of curves whose points are rationally equivalent to the Beauville-Voisin cycle c_S is dense. The pull-backs of these curves to A yield the conclusion. □

We are now ready to prove the main result of this section.

Theorem 4.4. *Let A be an abelian surface and $k \geq d \geq 0$ two integers. Let $Z \subset A^{(k)}$ an irreducible closed subvariety of dimension $k - d$ such that $\forall z, z' \in Z$ we have*

$$z + \iota(z) \equiv z' + \iota(z') \tag{4.2}$$

Then any $z \in Z$ lies in $S^k_d(A)$.

Proof. We argue as in [Voi15, Proof of Theorem 2.1] and we proceed by induction on k. The case $k = d$ is trivial, and the case $k = 1,\ d = 0$ is provided by Lemma 4.2. We consider the k-th cartesian product $A^{\times k} = A \times \cdots \times A$ and denote by $p := p_1$, respectively by $q := p_2 \times \cdots \times p_k$, the projection onto the first factor, respectively onto the last $(k-1)$-factors. Consider the natural morphism $\pi : A^{\times k} \longrightarrow A^{(k)}$ and let $\tilde{Z} := \pi^{-1}(Z)$. By abuse of notation we still denote by p and q the restrictions of these maps to $\tilde{Z}$. We will distinguish two cases.

Case 1: the map $p : \tilde{Z} \longrightarrow A$ is dominant.
Let $C \subset A$ be a curve such that $\forall p \in C$ we have $p + \iota(p) \equiv 2o_A$. Set $\tilde{Z}_C := p^{-1}(C)$. Notice that points of $\tilde{Z}_C$ are of the form

$$(c, z') : c \in C,\ c + z' \in Z. \tag{4.3}$$

In particular $c + \iota(c) \equiv 2o_A$.

We further distinguish two subcases.

Subcase 1.a: $\dim(q(\tilde{Z}_C)) = \dim(\tilde{Z}_C) = k - d - 1$.

From (4.3), (4.2) and the fact that $c + \iota(c) \equiv 2o_A$, for all $c \in C$ we deduce that

$$z' + \iota(z') \equiv z + \iota(z) - 2o_A$$

does not depend on the choice of $z' \in q(\tilde{Z}_C)$. In particular we can apply the inductive hypothesis to $q(\tilde{Z}_C)$ and deduce that $z' \in S_d^{k-1}(A)$, i.e.

$$z' + \iota(z') \equiv y' + 2(k-1-d)o_A \text{ with } y' \text{ effective}: \ \deg(y') = 2d.$$

Therefore for all $z \in Z_C$, $z = c + z'$ we have

$$z + \iota(z) = \big(c + \iota(c)\big) + \big(z' + \iota(z')\big) \equiv 2o_A + \big(y' + 2(k-1-d)o_A\big) = y' + 2(k-d)o_A$$

with y' effective such that $\deg(y') = 2d$.

Subcase 1.b: $\dim(q(\tilde{Z}_C)) < \dim(\tilde{Z}_C) = k-d-1$, for any curve C as above.
By Lemma 4.3, curves C such that $p + \iota(p) \equiv 2o_A$, $\forall p \in C$ are Zariski dense in A, therefore we can take one such curve and a point $z \in \tilde{Z}_C$ such that z' is general both in $\tilde{Z}_C$ and $\tilde{Z}$, and both varieties are smooth in z' of dimensions respectively $k-d-1$ and $k-d$. By hypothesis, the map q is not of maximal rank at z', which is general in $\tilde{Z}$. Hence, also $q(\tilde{Z})$ has dimension at most $k-d-1$. As all cycles z parametrized by $\tilde{Z}$ have $z + \iota(z)$ constant in $\mathrm{CH}_0(A)$, it follows that for any fiber F of q, all points c of $p(F)$ satisfy

$$c + \iota(c) = 2o_A.$$

However, this contradicts the fact that p is surjective, as the general point r of A does not satisfy $r + \iota(r) = 2o_A$.

Case 2: None of the projections onto the j-th factor, $j = 1, \dots k$, restricted to $\tilde{Z}$ is dominant. Notice that if one of these projections p_j were dominant up to exchanging the role of p_1 and p_j we would be in **Case 1**.

We set $C_j := \mathrm{Im}(p_j)$, if the image of the projection p_j on the j-th factor is one-dimensional, while we pick any curve C_j containing $\mathrm{Im}(p_j)$ if this image is a point. Consider now an ample symmetric curve $C \subset A$ such that $p + \iota(p) \equiv 2o_A$, $\forall p \in C$. The associated line bundle $L := p_1^*\mathcal{O}_A(C) \otimes \cdots p_k^*\mathcal{O}_A(C)$ is ample and so is its restriction to $C_1 \times \cdots \times C_k$. One sees that the (non-empty) intersection between $\tilde{Z}$ and the $(k-d)$-self intersection of L is

$$\tilde{Z} \cap \big[(k-d)! \sum_{i_1 < \cdots < i_{k-d}} p_{i_1}^* C \cdot \ldots \cdot p_{i_k}^* C\big].$$

Therefore for some $i_1 < \cdots < i_{k-d}$ we have

$$\tilde{Z} \cap p_{i_1}^* C \cdot \ldots \cdot p_{i_{k-d}}^* C \neq \emptyset.$$

An element in $\tilde{Z} \cap p_{i_1}^* C \cdots p_{i_{k-d}}^* C$ yields a cycle $z = \xi + w$ with $\xi \in C^{(k-d)}$ and $w \in A^{(d)}$. As a consequence

$$z + \iota(z) = \big(\xi + \iota(\xi)\big) + \big(w + \iota(w)\big) \equiv 2(k-d)o_A + y$$

with $y := w + \iota(w)$ effective 0-cycle of degree $2d$ and we are done. □

Proof of Theorem 1.1. The result follows immediately from Lemma 4.1 and Theorem 4.4. □

Proposition 4.5. *Let $z \in A^{(k)}$ be a point such that it is contained in the kernel of the summation map $A^{(k)} \longrightarrow A$. Suppose that $z \in S_d^k(A)$, $k \neq d$. Then O_z has dimension $k-d-1$.*

Proof. As z is in the kernel of the summation map, by [Blo76, Corollary 2.4] (see also [Bea86, Théorème]) we have $z \equiv \iota(z)$. As $z \in S_d^k(A)$, we have

$$2z \equiv z + \iota(z) \equiv \eta + \iota(\eta) + 2(k-d)o_A,$$

where η is effective and of degree d. Up to replacing η with a translate by a torsion point, we can suppose that also $\eta \in A^{(d)}$ is in the kernel of the summation map, hence $\iota(\eta) \equiv \eta$, from which we obtain that

$$z \equiv \eta + (k-d)o_A.$$

Therefore, to prove our claim it suffices to prove that $\dim(O_{(k-d)o_A}) = k-d-1$, which is the content of Lemma 4.6. □

Lemma 4.6. *Let A be an abelian surface. Then* $\dim(O_{jo_A}) = j-1$

Proof. Our claim is the content of [Lin16, Section 2.2]. First, we suppose that A is principally polarized and let C be a symmetric theta divisor passign through o_A. We construct the variety $V_{j,1}$ as in loc. cit. by taking the image of $C^{j+1} \longrightarrow A^{(j)}$ and restricting to the kernel of the summation map. Here the first map is $(a_0, \dots, a_j, a) \mapsto (a_0 +_A a, \dots, a_j +_A a)$. By [Lin16, Lemma 2.5], the class of a point in $V_{j,1}$ is constant. Clearly, $jo_A \in V_{n,1}$, hence our claim holds for principally polarized abelian varieties. The Chow ring is preserved under isogenies of abelian varieties by [Blo76] and all abelian surfaces are isogenous to a principally polarized abelian surface, hence our claim still holds. □

Remark 4.7. By definition $O_z \subset O_z^\iota$. Therefore, Theorem 1.1 implies a partial converse to Proposition 4.5: If we have an element $z \in A^{(k)}$ in the kernel of the summation map such that $\dim(O_z) \geq k-d-1$, we obtain that $z \in S^k_{d+1}(A)$. By Lemma 4.6, we expect that in this situation one has $\dim(O_z^\iota) = \dim(O_z)+1$.

In light of the above remark, we make the following:

Conjecture 4.8. Let A be an abelian surface and let $z \subset A^{(k)}$ be a cycle in the kernel of the summation map. Then $\dim(O_z^\iota) = \dim(O_z)+1$.

5. The second Chern classes of simple vector bundles

Given a Mukai vector $v = (r, l, s)$, we denote with $d(v) := \frac{l^2}{2} - rs + 1$. This number coincides with $\frac{1}{2}\dim(\mathrm{Ext}^1(F,F))$ for any stable sheaf F with Mukai vector v, that is half the dimension of the moduli space $M_H(v)$ (for any v generic polarization H). Note that $d(v) \geq 1$, since there are no spherical objects in $D^b(A)$ by [Bri08, Lemma 15.1]. We study here Conjecture 1.2. To motivate the restrictions on the first and second Chern class, let us consider the following examples:

Example 5.1. Let us take the Mukai vector $v = (1, 0, -1)$. Elements in the moduli space of stable sheaves with Mukai vector v can be described as the tensor product of the ideal sheaf $\mathcal{I}_p$ of a single point $p \in A$ with an element $L \in \mathrm{Pic}^0(A)$. We have $d(v) = 2$ and $c_2(\mathcal{I}_p) = p$. Therefore the ideal sheaf of a point lies in $S_1(A)$, or even in $S_0(A)$ if we have that $p+\iota(p) \equiv 2o_A$ (which by Theorem 2.4 holds on a dense subset of A). Tensoring with a degree zero line bundle L changes the second Chern class by a multiple of L^2, which is an element of $S_0(A)$ if L is symmetric, by Proposition 2.3.

To provide some evidence for the conjecture, we have the following:

Theorem 5.2. *Let A be an abelian surface and F a simple vector bundle of rank $r \geq 2$ on A with Mukai vector $v := v(F)$. Then*

$$c_2(F) \in S_{d(v)+r(r-2)}(A)$$

Remark 5.3. Notice that, if $r = 2$, this proves $c_2(F) \in S_{d(v)}(A)$ without any restriction on Chern classes. Therefore, assuming Conjecture 4.8, the above theorem actually gives $c_2(F) \in S_{d(v)-1}(A)$ for any sheaf with symmetric c_1 and c_2 which sums to zero, as in the proof of the theorem we obtain that $\dim(O_{c_2(F)}) \geq k - d(v)$.

As observed in [O'13], to prove the result for F on a K3 surface it is sufficient to prove it for a twist $F \otimes L$ by a line bundle L. In the abelian case, the situation is analogous:

Lemma 5.4. *Let A be an abelian surface and let F be a simple vector bundle on A with Mukai vector $v := v(F)$. Let L be a symmetric line bundle. Then*

$$c_2(F) \in S_j(A) \Leftrightarrow c_2(F \otimes L) \in S_j(A).$$

Proof. By a direct computation with Chern polynomials, we have

$$c_2(F\otimes L) = c_2(F) + c_1(F)c_1(L)(\mathrm{rk}(F)-1) + \frac{1}{2}c_1(L)^2(\mathrm{rk}(F)^2 - \mathrm{rk}(F)).$$

By the assumption on L, we have that $c_1(L)^2 \in S_0(A)$. If we consider $c_2(F\otimes L)+\iota(c_2(F\otimes L))$, we are left with the terms $c_2(F)+\iota(c_2(F))$ (which lies in $S_j(A)$ by hypothesis), $c_1(L)^2(\mathrm{rk}(F)^2 - \mathrm{rk}(F))$ (which lies in $S_0(A)$ by the assumption on L and $(c_1(F)+\iota c_1(F))c_1(L)(1-\mathrm{rk}(F))$. Now it suffices to notice that $c_1(F)+\iota(c_1(F)) = c_1(\det(F)\otimes\iota(\det(F))$, which is symmetric, hence also the last term lies in $S_0(A)$, and the claim follows. ☐

Hence, up to tensoring with a sufficiently ample symmetric line bundle we may assume that F is generated by its global sections and that $h^1(A, F^\vee) = 0$.

We recall the following results from [Voi15]. If $r := \mathrm{rk}(F)$ and $W \subset H^0(A, F)$ is a general vector subspace of dimension $(r-1)$, we consider the evaluation map

$$e_W : W \otimes \mathcal{O}_A \longrightarrow F.$$

Lemma 5.5. *The morphism e_W is generically injective, and the locus $Z \subset X$ where its rank is $< r-1$ consists of k distinct reduced points, where $k = c_2^{top}(F)$.*

Following [Voi15], the above lemma gives us a rational map $\phi : G(r-1, H^0(A, F)) \dashrightarrow A^{(k)}$ which sends a subspace W to the subset Z where e_W is not injective.

Proposition 5.6 (Proposition 3.2, [Voi15])**.** *If F is simple and satisfies $h^1(A, F^\vee) = h^1(A, F) = 0$, the rational map ϕ is generically one to one on its image.*

With this result, we can now proceed to prove the main theorem of the section:

Proof of Theorem 5.2. Let F be a simple vector bundle and let $v(F) := (r, l, s)$, where $r = \mathrm{rk}(F)$, $l = c_1^{top}(F)$. By Lemma 5.4, we can suppose that F is globally generated and $H^1(A, F^\vee) = 0$. We now follow very closely [Voi15, Proof of Theorem 1.9]: we have that $\chi(A, \mathrm{End}(F)) = (v, v^\vee) = 2rs - l^2$ and, by Riemann-Roch,

$$\chi(A, \mathrm{End}(F)) = (r-1)l^2 - 2rc_2^{top}(F).$$

This gives $s = \frac{l^2}{2} - c_2^{top}(F)$. We set $k = c_2^{top}(F)$. By a direct Chern class computation, we have that $\chi(A, F) = s$. As F is globally generated and has trivial H^1, this number equals $h^0(A, F)$. Proposition 5.6 now gives a rational map from $G(r-1, h^0(A, F))$ to $A^{(k)}$ whose image is given by a constant cycle subvariety.

Its dimension is $(r-1)(s-r+1) = rs+2r-s-1-r^2$. By Theorem 4.4, it follows that $c_2(F)$ lies in $S_d^k(A)$, where $d = d(v)+r(r-2)$ and it is obtained by computing $k-d = \dim(G(r-1, s))$. ☐

Corollary 5.7. *Assume Conjecture 1.2. Let $v \in H^*(A, \mathbb{Z})$ be a Mukai vector. Assume there exists a simple vector bundle F on A with Mukai vector v. Then*

$$S_d^k(A) = \{c_2(G) : G \text{ simple vector bundle on } A \text{ with Mukai vector } v, \text{ whose } c_1 \text{ is symmetric}\}$$

with $d = d(v) - 1,\ k = c_2(v) = c_2^{top}(F)$.

Proof. The inclusion "$\supset$" is the content of Conjecture 1.2. For the other inclusion we argue *mutatis mutandi* as in [Voi15, Proof of Corollary 3.4]. We repeat it here for greater clarity. Let

$$B := \{c_2(G) : G \text{ simple vector bundle on } A \text{ with } v(G) = v \text{ and symmetric } c_1(G)\}.$$

Let $cl : A^{(d)} \longrightarrow \mathrm{CH}_0(A)$ be the cycle class map. We start by claiming that there is a dense open subset U such that

$$cl(U) + cl(\iota(U)) + 2(k-d)o_A \subset B. \tag{5.1}$$

As Y is simple and has symmetric c_1, we can find a smooth quasi projective variety Y, a point $y_0 \in Y$ and a locally free sheaf $\mathcal{F}$ on $Y \times A$ such that $F \cong \mathcal{F}_{y_0}$, the Kodaira spencer map is injective and all sheaves parametrized by Y are simple with symmetric c_1. Let $\Gamma := c_2(\mathcal{F}) \in \mathrm{CH}^2(Y \times A)$ and consider the following set $R \subset Y \times A^{(d)}$:

$$R := \{(y,z),\ \Gamma_*(y) + \iota(\Gamma_*(y)) \equiv c_2(\mathcal{F}_y) + \iota(c_2(\mathcal{F}_y)) \equiv cl(z) + \iota(cl(z)) + 2(k-d)o_A \in \mathrm{CH}_0(A)\},$$

where $k = c_2^{top}(F)$. The first projection is surjective by Conjecture 1.2. Let R_0 be a component of R which dominates the first factor. The second projection $R_0 \longrightarrow A^{(d)}$ is also dominant. Indeed, using Mumford's theorem as in the case of $K3$ surface one sees that (5.1) holds. Then we conclude by invoking Theorem 2.4. ☐

6. From simple vector bundles to stable sheaves

In this section we explain the first consequence of Conjecture 1.2.

Theorem 6.1. *Let A be an abelian surface and H a fixed polarization. Assume that Conjecture 1.2 holds. Then for every H-slope stable torsion free sheaf F on A such that $c_1(F)$ is symmetric, we have*

$$c_2(F) \in S_{d(F)-1}(A),$$

where $d(F) := \frac{1}{2}\dim \mathrm{Ext}^1(F,F)$.

The argument follows the strategy of [SYZ20, Proposition 1.7] with some modifications to adapt it to this context. We first need the following two lemmas.

Lemma 6.2. *Let*

$$E \longrightarrow F \longrightarrow G \longrightarrow E[1] \tag{6.1}$$

be a distinguished triangle in $D^b(A)$. If two of $c_2(E)$, $c_2(F)$, $c_2(G)$ lie in $S_i(A)$ and $S_j(A)$ respectively, then the third lies in $S_{i+j}(A)$.

Proof. This is the analogue of [SYZ20, Corollary 1.2]. From (6.1) we deduce that

$$c_2(F) = c_2(E) + c_2(G) + D,$$

where D is spanned by intersections of divisor classes. Taking the involution on both sides we get

$$c_2(F) + \iota(c_2(F)) = [c_2(E) + \iota(c_2(E))] + [c_2(G) + \iota(c_2(G))] + [D + \iota(D)].$$

Note that $D + \iota(D)$ lies in the image of the intersection map restricted to symmetric classes

$$\mathrm{Pic}(A)^\iota \otimes Pic(A)^\iota \longrightarrow \mathrm{CH}_0(A),\ L \otimes M \mapsto L \cdot M.$$

Hence the conclusion follows from Proposition 2.3 and Remark 3.2. ☐

Lemma 6.3. *Let*

$$E \longrightarrow F \longrightarrow G \longrightarrow E[1]$$

be a distinguished triangle in $D^b(A)$ such that $\mathrm{Hom}(E,G) = 0$. If $c_2(E) \in S_{d(E)}(A)$ and $c_2(G) \in S_{d(G)-1}(A)$, then

$$c_2(F) \in S_{d(F)-1}(A).$$

Proof. By Mukai's Lemma [BB17, Lemma 2.5] we have

$$d(E) + d(G) \leq d(F).$$

Combining the assumption, Lemma 6.2 and Lemma 3.3, we conclude that

$$c_2(F) \in S_{d(E)+d(G)-1}(A) \subset S_{d(F)-1}(A).$$

☐

Proof of Theorem 6.1. Consider the short exact sequence

$$0 \longrightarrow F \longrightarrow F^{\vee\vee} \longrightarrow Q \longrightarrow 0,$$

where $F^{\vee\vee}$ is a vector bundle and Q is a torsion sheaf supported on points. Since $c_1(F) = c_1(F^{\vee\vee})$ is symmetric, Conjecture 1.2 implies that $c_2(F^{\vee\vee}) \in S_{d(F^{\vee\vee})-1}$. On the other hand, if d is the lenght of the support of Q, then $d(Q) \geq d$. Hence $c_2(Q) \in S_d(A) \subset S_{d(Q)}(A)$. Applying Lemma 6.3 to the distinguished triangle

$$Q[-1] \longrightarrow F \longrightarrow F^{\vee\vee} \longrightarrow Q,$$

which satisfies $\mathrm{Hom}(Q[-1], F^{\vee\vee}) = 0$ since $F^{\vee\vee}$ is locally free and Q is supported in dimension 0, we deduce the statement. □

We point out the following corollary of Theorem 6.1.

Corollary 6.4. *Let E be an iterated extension of a slope stable torsion free sheaf F with $c_1(F)$ symmetric. If Conjecture 1.2 holds, then*

$$c_2(E) \in S_{d(E)-1}(A).$$

Proof. This is the analogue of [SYZ20, Proposition 1.8]. Note that $c_1(E) = mc_1(F)$ is symmetric and $c_2(E) = mc_2(F) + D$ for some positive integer m, where D is the intersection of symmetric divisors. By Proposition 2.3 it follows that $D \in S_0(A)$. Thus by Theorem 6.1 and Corollary 3.5 it follows that $c_2(E) \in S_{d(F)-1}(A)$. The conclusion follows from

$$2d(E) = v(E)^2 + 2\dim \mathrm{Hom}(E, E) = m^2 v(F)^2 + 2\dim \mathrm{Hom}(E, E) \geq v(F)^2 + 2 = 2d(F).$$

□

Remark 6.5. We expect that Theorem 6.1 holds for objects in $D^b(A)$ with symmetric first Chern class. However, the latter assumption does not allow us to use the induction argument as in [SYZ20].

Remark 6.6. Let T be a torsion sheaf supported on a symmetric curve $\tilde{C}$. Then by Lemma 3.4 we have $c_2(T) \in S_g(A)$, where g is the genus of $\tilde{C}/\iota$. On the other hand, the genus $\tilde{g}$ of $\tilde{C}$ satisfies

$$\tilde{g} = 1 + \tilde{C}^2 \leq \frac{1}{2} v(T)^2 + \dim \mathrm{Hom}(T, T) = d(T)$$

and

$$\tilde{g} = 2g - 1 + \frac{B}{2}.$$

Thus

$$d(T) - 1 \geq \tilde{g} - 1 = 2g - 2 + \frac{B}{2} \geq g.$$

The last equality follows from the fact that if $g < 2 - \frac{B}{2}$, then $\tilde{g} \leq 1$ which is impossible on A. It follows that $c_2(T) \in S_{d(T)-1}(A)$.

7. Coisotropic subvarieties

Let A be an abelian surface with a fixed polarization H. Let $v = (r, \ell, s) \in H^0(A, \mathbb{Z}) \oplus NS(A) \oplus H^4(A, \mathbb{Z})$ be a primitive and positive Mukai vector, i.e. either $r > 0$, or if $r = 0$, ℓ is effective and $s \neq 0$, or $r = \ell = 0$ and $s < 0$. We denote by $M_H(v)$ the moduli space of H-semistable sheaves on A with Chern class v, and let $M_H(v)^{\mathrm{st}}$ be its stable locus. If H is v-generic, then $M_H(v) = M_H(v)^{\mathrm{st}}$. Let $\Phi\colon D^b(A) \longrightarrow D^b(A^\vee)$ be the Fourier-Mukai transform associated to the Poincaré line bundle $\mathcal{P}$ on $A \times A^\vee$, and consider the morphism

$$alb_v : M_H(v) \longrightarrow A \times A^\vee,\ F \mapsto (det(\Phi(F)) \otimes det(\Phi(F_0^\vee)), det(F) \otimes det(F_0^\vee)),$$

where $F_0 \in M_H(v)$ is fixed. Denote by $K_H(v)$ a fiber of alb_v.

Theorem 7.1 ([Yos01], Theorems 0.1 and 0.2)**.** *Assume that $v^2 \geq 6$ and H is v-generic. Then alb_v is the Albanese morphism and $K_H(v)$ is a projective hyperkäheler manifold of dimension v^2-2 deformation equivalent to a generalized Kummer variety.*

An important consequence of Conjecture 1.2 is the existence of algebraically coisotropic subvarieties of the form

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, dashed, "p"] & K_H(v) \\ B & \end{tikzcd}$$

where Y has codimension i and the general fibers of p are constant cycle subvarieties of $K_H(v)$ of dimension i.

Let us first make the following observations.

Lemma 7.2. *Assume that $v^2 \geq 6$ and H is v-generic. Let $\alpha : M_H(v) \longrightarrow A$ be the map sending a sheaf F to the sum of $c_2(F)$. Then there exists a choice of $F_0 \in M_H(v)$ such that $K_H(v)$ is contained in a fiber of α.*

Proof. By fixing a Mukai vector we fix the integer $s := deg(c_2(\mathcal{F}))$ for all sheaves $F \in M_H(v)$. We therefore have the actions of A on $M_H(v)$ and on A itself which are given by

$$A \times M_H(v) \longrightarrow M_H(v),\ (a, F) \mapsto t_a(F)$$

and

$$A \times A \longrightarrow A,\ (a, b) \mapsto (b +_A sa).$$

The morphism α is equivariant with respect to this action, hence α is an isotrivial fibration with isomorphic fibers. By universality of the Albanese map, all fibers of alb_v are contained in fibers of α and, as the Albanese fibration is isotrivial by an analogous argument, we can choose a sheaf F_0 to ensure that $K_H(v) \subset \alpha^{-1}(0)$. □

Proposition 7.3. *Let E, F be two objects in $K_H(v)$. Then $[E] = [F]$ in $\mathrm{CH}_0(K_H(v))$ if and only if $c_2(E) = c_2(F)$.*

Proof. The proof given in [MZ20, Theorem] works as well for moduli spaces on abelian surfaces. To pass to the fiber over zero of the Albanese morphism we observe that two rationally equivalent sheaves in the moduli space must lie in fibers differing by torsion points. See also [CLZ26, Theorem 6.10]. □

We are now ready to prove the main result of this section.

Proof of Theorem 1.3. We argue as in [SYZ20, Theorem 0.5]. Set $M := M_H(v)$ and $d := d(v)-1$. We consider the Hilbert scheme $A^{[d]}$ of d-points on A and the following correspondence

$$\overline{R} := \{(\mathcal{E}, \xi) \in M \times A^{[d]} : c_2(\mathcal{E}) + i(c_2(\mathcal{E})) \equiv \mathrm{supp}(\xi) + \iota(\mathrm{supp}(\xi)) + 2m \cdot o_A, \exists m \in \mathbb{Z}\}.$$

Set $K := K_H(v)$ and $\mathrm{Kum} := \mathrm{Kum}_{d-1}(A)$. The main point is that by Lemma 7.2 and [Blo76, Corollary 2.4] the correspondence induces the following correspondence

$$R := \{(E, \xi) \in K \times \mathrm{Kum} : c_2(E) \equiv \mathrm{supp}(\xi) + m \cdot o_A, \exists m \in \mathbb{Z}\}.$$

We denote by p_K and p_{Kum} the projections of R on the two factors. We start by analyzing the fibers of p_K and p_{Kum}. By definition of R it follows that for any $(E, \xi), (E, \xi') \in R$ we have

$$\mathrm{supp}(\xi) \equiv \mathrm{supp}(\xi'). \tag{7.1}$$

Analogously, if $(E, \xi), (E', \xi) \in R$, then $c_2(E) \equiv c_2(E')$. Proposition 7.3 implies that $[E] \equiv [E'] \in \mathrm{CH}_0(K)$.

Now by Theorem 6.1 the projection p_K is dominant. Moreover, Conjecture 1.2 impies that the projection p_{Kum} is dominant, using that for every $\ell \in \mathrm{NS}(A)$ we can find a symmetric line bundle corresponding to it.

Fix one component R_0 of R dominating both factors. We restrict to the non-empty open subsets $U \subset K$ (respectively $V \subset \mathrm{Kum}$) over which the projection p_K (resp. p_{Kum}) is surjective and finite.

Consider a curve C such that $\forall p \in C$ we have $p + \iota(p) \equiv 2o_A$. Let $V_{i,1}(C)$ be the subvariety of Kum associated with C (using the construction [Lin16, Section 2.2], see also the proof of Lemma 4.6). We set

$$Z := V_{i,1}(C) \times_{\mathrm{Kum}} R_0$$

and $\psi : Z \dashrightarrow R_0$ the natural induced map. By Lemma 4.3, as we let the curve C vary among symmetric hyperelliptic curves, we may assume that $\psi(Z) \cap p_K^{-1}(U) \neq \emptyset$. Then the subset

$$p_K(\psi(Z))$$

parametrizes objects $\mathcal{E} \in K$ such that $c_2(\mathcal{E})$ is constant in $\mathrm{CH}_0(A)$.

□

We finish this section with a comparison with [CLZ26, Conjecture 7.2]

Proposition 7.4. *Let A be an abelian surface. Assume Conjecture 4.8. Then the following are equivalent:*

(1) *Conjecture 1.2 holds.*
(2) *Conjecture* 7.2 *of* [CLZ26] *holds.*

Proof. Let F be a sheaf in a moduli space $M_H(v)$. Let us suppose in addition that F lies in a fiber of the Albanese map such that $c_1(F)$ is symmetric and $c_2(F)$ sums to zero. Then, by [Blo76, Corollary 2.4], we have that $c_2(F) \equiv \iota(c_2(F))$, thus $c_2(F)$ and $c_2(F)_{(2)}$ (in the notation of [CLZ26]) differ by a multiple of o_A, which does not change the stratum $S_i(A)$ of the filtration. Let k be the degree of $c_2(F)$. By Theorem 1.1 and Proposition 7.3, the rational equivalence orbit of F in $K_H(v)$ is of dimension at least $k - d(v)$ if and only if $c_2(F) \in S^k_{d(v)-1}(A)$, which gives the equivalence between the two conjectures.

□

References

[BB17] Arend Bayer and Tom Bridgeland. Derived automorphism groups of K3 surfaces of Picard rank 1. *Duke Math. J.*, 166(1):75–124, 2017. – cited on p. 9

[Bea86] Arnaud Beauville. Sur l'anneau de Chow d'une variété abélienne. *Math. Ann.*, 273(4):647–651, 1986. – cited on p. 2, 6

[BL13] Christina Birkenhake and Herbert Lange. *Complex abelian varieties*. Springer Science & Business Media, 2013. – cited on p. 2

[Blo76] Spencer Bloch. Some elementary theorems about algebraic cycles on abelian varieties. *Inventiones mathematicae*, 37(3):215–228, 1976. – cited on p. 6, 7, 11, 12

[Bri08] Tom Bridgeland. Stability conditions on $K3$ surfaces. *Duke Math. J.*, 141(2):241–291, 2008. – cited on p. 7

[BV04] Arnaud Beauville and Claire Voisin. On the Chow ring of a $K3$ surface. *J. Algebraic Geom.*, 13(3):417–426, 2004. – cited on p. 1

[CGL22] Xi Chen, Frank Gounelas, and Christian Liedtke. Curves on K3 surfaces. *Duke Mathematical Journal*, 171(16):3283–3362, 2022. – cited on p. 2, 3

[CLZ26] Zaiyuan Chen, Zhiyuan Li, and Ruxuan Zhang. Bloch's conjecture for equivalences between twisted abelian surfaces and applications. *arXiv preprint arXiv:2606.22323*, 2026. – cited on p. 2, 11, 12

[Huy10] Daniel Huybrechts. Chow groups of k3 surfaces and spherical objects. *Journal of the European Mathematical Society*, 12(6):1533–1551, 2010. – cited on p. 1

[Lin16] Hsueh-Yung Lin. On the chow group of zero-cycles of a generalized kummer variety. *Advances in Mathematics*, 298:448–472, 2016. – cited on p. 7, 12

[Mac04] Catriona Maclean. Chow groups of surfaces with $h^{2,0} \leq 1$. *Comptes Rendus. Mathématique*, 338(1):55–58, 2004. – cited on p. 3

[MZ20] Alina Marian and Xiaolei Zhao. On the group of zero-cycles of holomorphic symplectic varieties. *Épijournal de Géométrie Algébrique*, 4, 2020. – cited on p. 11

[O'13] Kieran G O'Grady. Moduli of sheaves and the Chow group of K3 surfaces. *Journal de mathématiques pures et appliquées*, 100(5):701–718, 2013. – cited on p. 1, 3, 7

[SYZ20] Junliang Shen, Qizheng Yin, and Xiaolei Zhao. Derived categories of surfaces, o'grady's filtration, and zero-cycles on holomorphic symplectic varieties. *Compositio Mathematica*, 156(1):179–197, 2020. – cited on p. 1, 9, 10, 11

[Voi15] Claire Voisin. Rational equivalence of 0-cycles on k3 surfaces and conjectures of Huybrechts and O'Grady. *Recent advances in algebraic geometry*, 417:422–36, 2015. – cited on p. 1, 3, 5, 8

[Voi16] Claire Voisin. Remarks and questions on coisotropic subvarieties and 0-cycles of hyper-Kähler varieties. In *K3 surfaces and their moduli*, volume 315 of *Progr. Math.*, pages 365–399. Birkhäuser/Springer, [Cham], 2016. – cited on p. 2

[Yos01] Kōta Yoshioka. Moduli spaces of stable sheaves on abelian surfaces. *Mathematische Annalen*, 321(4):817–884, 2001. – cited on p. 11

Giovanni Mongardi, Alma Mater Studiorum, Università di Bologna, P.zza di porta san Donato 5, 40126 Bologna, Italia
Email address: giovanni.mongardi2@unibo.it

Gianluca Pacienza, Université de Lorraine, CNRS, IECL, F-54000 Nancy – France
Email address: gianluca.pacienza@univ-lorraine.fr

Laura Pertusi, Università degli studi di Milano, via Cesare Saldini 50, 20133 Milano, Italia
Email address: laura.pertusi@unimi.it